Zadania 3D

Zadanie 1.

Niech $H = 1 + i + j + k$. Wyznacz **a)** $\boldsymbol{H^{(-1)}}$; **b)** H^2 **c)** $\boldsymbol{H^3}$

Rozwiązanie:

a)

$$H^{(-1)} = \frac{1}{H} = \frac{1}{1 + i + j + k}$$

$$\overline{H} = 1 - i - j - k$$

$$|H| = 1 + 1 + 1 + 1 = 4$$

$$H^{(-1)} = \frac{1 - i - j - k}{4} = \frac{1}{4} - \frac{i}{4} - \frac{j}{4} - \frac{k}{4}$$

b)

$$H^2 = H \cdot H = (1 + i + j + k) \cdot (1 + i + j + k) =$$

$$= 1 + i + j + k + i - 1 + ij + ik + j + ji - 1 + jk + k + ki + kj - 1 =$$

$$= -2 + 2i + 2j + 2k + k - j - k + i + j - i = -2 + 2i + 2j + 2k$$

c)

$$H^3 = H^2 \cdot H = (-2 + 2i + 2j + 2k) \cdot (1 + i + j + k) =$$

$$= -2 - 2i - 2j - 2k + 2i - 2 + 2ij + 2ik + 2j + 2ji - 2 + 2jk + 2k + 2ki + 2kj - 2 =$$

$$= -8 + 2k - 2j - 2k + 2i + 2j - 2i = -8$$

Zadanie 2.

Punkt $A = (1; 2; 3)$ obróć o kąt $90°$ dookoła osi *k.*

Rozwiązanie:

$$h_A = i + 2j + 3k$$

$$v = k$$

$$\varphi = 90°$$

$$\frac{\varphi}{2} = 45°$$

$$a = \cos 45° = \frac{\sqrt{2}}{2}; \qquad d = \sin 45° = \frac{\sqrt{2}}{2}$$

$$q = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}k$$

$$q^{-1} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}k$$

$$A' = q \cdot h_A \cdot q^{-1} = \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}k\right) \cdot (i + 2j + 3k) \cdot \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}k\right) =$$

$$= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}k\right) \cdot \left(\frac{\sqrt{2}}{2}i - \frac{\sqrt{2}}{2}ik + \sqrt{2}j - \sqrt{2}jk + \frac{3\sqrt{2}}{2}k + \frac{3\sqrt{2}}{2}\right) =$$

$$= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}k\right) \cdot \left(\frac{3\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i + \sqrt{2}j + \frac{3\sqrt{2}}{2}k + \frac{\sqrt{2}}{2}j - \sqrt{2}i\right) =$$

$$= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}k\right) \cdot \left(\frac{3\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i + \frac{3\sqrt{2}}{2}j + \frac{3\sqrt{2}}{2}k\right) =$$

$$= \frac{3}{2} - \frac{1}{2}i + \frac{3}{2}j + \frac{3}{2}k + \frac{3}{2}k - \frac{1}{2}ki + \frac{3}{2}kj - \frac{3}{2} = -\frac{1}{2}i + \frac{3}{2}j + 3k - \frac{1}{2}j - \frac{3}{2}i =$$

$$= -2i + j + 3k$$

$$A' = (-2; 1; 3)$$

Zadanie 3.

Niech $A = (0; 0; 4)$ $\quad i \quad$ $B = (3; 0; 0)$. Odcinek *AB* obróć o kąt $60°$ dookoła osi j.

Rozwiązanie:

$$h_A = 4k$$

$$v = j$$

$$\varphi = 60°$$

$$\frac{\varphi}{2} = 30°$$

$$a = \cos 30° = \frac{\sqrt{3}}{2}; \qquad c = \sin 30° = \frac{1}{2}$$

$$q = \frac{\sqrt{3}}{2} + \frac{1}{2}j$$

$$q^{-1} = \frac{\sqrt{3}}{2} - \frac{1}{2}j$$

$$A' = q \cdot h_A \cdot q^{-1} = \left(\frac{\sqrt{3}}{2} + \frac{1}{2}j\right) \cdot 4k \cdot \left(\frac{\sqrt{3}}{2} - \frac{1}{2}j\right) = \left(\frac{\sqrt{3}}{2} + \frac{1}{2}j\right) \cdot \left(2\sqrt{3}k - 2kj\right) =$$

$$= \left(\frac{\sqrt{3}}{2} + \frac{1}{2}j\right) \cdot \left(2\sqrt{3}k + 2i\right) = 3k + \sqrt{3}i + \sqrt{3}jk + ji = \sqrt{3}i + 3k + \sqrt{3}i - k = 2\sqrt{3}i + 2k$$

$$A' = \left(2\sqrt{3}; 0; 2\right)$$

$$h_B = 3i$$

$$v = j$$

$$\varphi = 60°$$

$$\frac{\varphi}{2} = 30°$$

$$a = \cos 30° = \frac{\sqrt{3}}{2}; \qquad c = \sin 30° = \frac{1}{2}$$

$$q = \frac{\sqrt{3}}{2} + \frac{1}{2}j$$

$$q^{-1} = \frac{\sqrt{3}}{2} - \frac{1}{2}j$$

$$B' = q \cdot h_A \cdot q^{-1} = \left(\frac{\sqrt{3}}{2} + \frac{1}{2}j\right) \cdot 3i \cdot \left(\frac{\sqrt{3}}{2} - \frac{1}{2}j\right) = \left(\frac{\sqrt{3}}{2} + \frac{1}{2}j\right) \cdot \left(\frac{3\sqrt{3}}{2}i - \frac{3}{2}ij\right) =$$

$$= \left(\frac{\sqrt{3}}{2} + \frac{1}{2}j\right) \cdot \left(\frac{3\sqrt{3}}{2}i - \frac{3}{2}k\right) = \frac{9}{4}i - \frac{3\sqrt{3}}{4}k + \frac{3\sqrt{3}}{4}ji - \frac{3}{4}jk = \frac{9}{4}i - \frac{3\sqrt{3}}{4}k - \frac{3\sqrt{3}}{4}k - \frac{3}{4}i$$

$$=$$

$$= \frac{3}{2}i - \frac{3\sqrt{3}}{2}k$$

$$B' = \left(\frac{3}{2}; 0; -\frac{3\sqrt{3}}{2}\right)$$

Zadanie 4.

Niech $A = (1; 2; 0); B = (3; 5; 0); C = (3; -5; 0)$. Trójkąt *ABC* obróć o kąt $180°$ dokoła prostej, zawierającej punkty *O* i *D*, jeśli $O = (0; 0; 0) \quad i \quad D = (3; 0; 3)$.

Uwaga!!!

Błąd w treści zadania.

Rozwiązanie:

$$h_A = i + 2j$$

$$v = \frac{\sqrt{2}}{2}i + \frac{\sqrt{2}}{2}k$$

$$\varphi = 180°$$

$$\frac{\varphi}{2} = 90°$$

$$a = \cos 90° = 0; \qquad b = \frac{\sqrt{2}}{2} \cdot \sin 90° = \frac{\sqrt{2}}{2}; \qquad d = \frac{\sqrt{2}}{2} \cdot \sin 90° = \frac{\sqrt{2}}{2}$$

$$q = \frac{\sqrt{2}}{2} i + \frac{\sqrt{2}}{2} k$$

$$q^{-1} = -\frac{\sqrt{2}}{2} i - \frac{\sqrt{2}}{2} k$$

$$A' = \left(\frac{\sqrt{2}}{2} i + \frac{\sqrt{2}}{2} k\right) \cdot (i + 2j) \cdot \left(-\frac{\sqrt{2}}{2} i - \frac{\sqrt{2}}{2} k\right) =$$

$$= \left(\frac{\sqrt{2}}{2} i + \frac{\sqrt{2}}{2} k\right) \cdot \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} ik - \sqrt{2} ji - \sqrt{2} jk\right) =$$

$$= \left(\frac{\sqrt{2}}{2} i + \frac{\sqrt{2}}{2} k\right) \cdot \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} j + \sqrt{2} k - \sqrt{2} i\right) =$$

$$= \left(\frac{\sqrt{2}}{2} i + \frac{\sqrt{2}}{2} k\right) \cdot \left(\frac{\sqrt{2}}{2} - \sqrt{2} i + \frac{\sqrt{2}}{2} j + \sqrt{2} k\right) = \frac{1}{2} i + 1 + \frac{1}{2} ij + ik + \frac{1}{2} k - ki + \frac{1}{2} kj - 1 =$$

$$= \frac{1}{2} i + 1 + \frac{1}{2} k - j + \frac{1}{2} k - j - \frac{1}{2} i - 1 = k - 2j$$

$$A' = (0; -2; 1)$$

$$h_B = 3i + 5j$$

$$v = \frac{\sqrt{2}}{2} i + \frac{\sqrt{2}}{2} k$$

$$\varphi = 180°$$

$$\frac{\varphi}{2} = 90°$$

$$a = \cos 90° = 0; \qquad b = \frac{\sqrt{2}}{2} \cdot \sin 90° = \frac{\sqrt{2}}{2}; \qquad d = \frac{\sqrt{2}}{2} \cdot \sin 90° = \frac{\sqrt{2}}{2}$$

$$q = \frac{\sqrt{2}}{2} i + \frac{\sqrt{2}}{2} k$$

$$q^{-1} = -\frac{\sqrt{2}}{2} i - \frac{\sqrt{2}}{2} k$$

$$B' = \left(\frac{\sqrt{2}}{2} i + \frac{\sqrt{2}}{2} k\right) \cdot (3i + 5j) \cdot \left(-\frac{\sqrt{2}}{2} i - \frac{\sqrt{2}}{2} k\right) =$$

$$= \left(\frac{\sqrt{2}}{2}i + \frac{\sqrt{2}}{2}k\right) \cdot \left(\frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}ik - \frac{5\sqrt{2}}{2}ji - \frac{5\sqrt{2}}{2}jk\right) =$$

$$= \left(\frac{\sqrt{2}}{2}i + \frac{\sqrt{2}}{2}k\right) \cdot \left(\frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}j + \frac{5\sqrt{2}}{2}k - \frac{5\sqrt{2}}{2}i\right) =$$

$$= \left(\frac{\sqrt{2}}{2}i + \frac{\sqrt{2}}{2}k\right) \cdot \left(\frac{3\sqrt{2}}{2} - \frac{5\sqrt{2}}{2}i + \frac{3\sqrt{2}}{2}j + \frac{5\sqrt{2}}{2}k\right) =$$

$$= \frac{3}{2}i + \frac{5}{2} + \frac{3}{2}ij + \frac{5}{2}ik + \frac{3}{2}k - \frac{5}{2}ki + \frac{3}{2}kj - \frac{5}{2} = \frac{3}{2}i + \frac{5}{2} + \frac{3}{2}k - \frac{5}{2}j + \frac{3}{2}k - \frac{5}{2}j - \frac{3}{2}i - \frac{5}{2}$$

$$=$$

$$= -5j + 3k$$

$$B' = (0;\ -5; 3)$$

$$h_C = 3i - 5j$$

$$v = \frac{\sqrt{2}}{2}i + \frac{\sqrt{2}}{2}k$$

$$\varphi = 180°$$

$$\frac{\varphi}{2} = 90°$$

$$a = \cos 90° = 0; \qquad b = \frac{\sqrt{2}}{2} \cdot \sin 90° = \frac{\sqrt{2}}{2}; \qquad d = \frac{\sqrt{2}}{2} \cdot \sin 90° = \frac{\sqrt{2}}{2}$$

$$q = \frac{\sqrt{2}}{2}i + \frac{\sqrt{2}}{2}k$$

$$q^{-1} = -\frac{\sqrt{2}}{2}i - \frac{\sqrt{2}}{2}k$$

$$C' = \left(\frac{\sqrt{2}}{2}i + \frac{\sqrt{2}}{2}k\right) \cdot (3i - 5j) \cdot \left(-\frac{\sqrt{2}}{2}i - \frac{\sqrt{2}}{2}k\right) =$$

$$= \left(\frac{\sqrt{2}}{2}i + \frac{\sqrt{2}}{2}k\right) \cdot \left(\frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}ik + \frac{5\sqrt{2}}{2}ji + \frac{5\sqrt{2}}{2}jk\right) =$$

$$= \left(\frac{\sqrt{2}}{2}i + \frac{\sqrt{2}}{2}k\right) \cdot \left(\frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}j - \frac{5\sqrt{2}}{2}k + \frac{5\sqrt{2}}{2}i\right) =$$

$$= \left(\frac{\sqrt{2}}{2}i + \frac{\sqrt{2}}{2}k\right) \cdot \left(\frac{3\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}i + \frac{3\sqrt{2}}{2}j - \frac{5\sqrt{2}}{2}k\right) =$$

$$= \frac{3}{2}i + \frac{5}{2} + \frac{3}{2}ij - \frac{5}{2}ik + \frac{3}{2}k + \frac{5}{2}ki + \frac{3}{2}kj - \frac{5}{2} = \frac{3}{2}i + \frac{5}{2} + \frac{3}{2}k + \frac{5}{2}j + \frac{3}{2}k + \frac{5}{2}j - \frac{3}{2}i - \frac{5}{2}$$

$$=$$

$$= 5j + 3k$$

$$C' = (0;\ 5; 3)$$