Zadania zespolone

Zadanie 1.

Wykonaj podane działania:

a) $(-2+3i)+(7-8i)$; b) $(4i-3)-(1+10i)$; c) $(\sqrt{2}+i)\cdot(3-\sqrt{3}i)$; d) $\frac{2-3i}{5+4i}$

Rozwiązanie

a) $(-2+3i)+(7-8i)=5-5i$

b) $(4i-3)-(1+10i)=-4-6i$

c) $(\sqrt{2}+i)\cdot(3-\sqrt{3}i)=3\sqrt{2}-\sqrt{6}i+3i+\sqrt{3}=3\sqrt{2}+\sqrt{3}+(3-\sqrt{6})i$

d) $\frac{2-3i}{5+4i}=\frac{(2-3i)(5-4i)}{(5+4i)(5-4i)}=\frac{10-8i-15i-12}{25-20i+20i+16}=\frac{-2-23i}{41}=-\frac{2}{41}-\frac{23}{41}i$

Zadanie 2.

Oblicz moduły podanych liczb zespolonych

a) $4i$ b) $12i-5$ c) $\sqrt{7}+\sqrt{29}i$ d) $(\sqrt{5}-\sqrt{3})+(\sqrt{5}+\sqrt{3})i$ e) $\sin\alpha+i\cos\alpha$

Rozwiązanie

a) $|4i|=4$

b) $|12i-5|=\sqrt{12^2+5^2}=\sqrt{144+25}=\sqrt{165}$

c) $|\sqrt{7}+\sqrt{29}i|=\sqrt{\left(\sqrt{7}\right)^2+\left(\sqrt{29}\right)^2}=\sqrt{7+29}=\sqrt{36}=6$

d) $|\,(\sqrt{5}-\sqrt{3})+(\sqrt{5}+\sqrt{3})i|=\sqrt{\left(\sqrt{5}-\sqrt{3}\right)^2+\left(\sqrt{5}+\sqrt{3}\right)^2}=$

$$=\sqrt{5-2\sqrt{15}+3+5+2\sqrt{15}+3}=\sqrt{16}=4$$

e)|) $\sin\alpha+i\cos\alpha|=\sqrt{\sin^2\alpha+\cos^2\alpha}=\sqrt{1}=1$

Zadanie 3.

Zapisz podane liczby w postaci trygonometrycznej

a) $-\sqrt{5}$ b) $-6+6i$ c) $-2i$ d) $\sqrt{3}+i$ e) $\sin\alpha-i\cos\alpha$ f) $1-i\cot\alpha$ g) $1+\cos\alpha+i\sin\alpha$

Rozwiązanie

a) $\boldsymbol{-\sqrt{5} = \sqrt{5}(\cos\pi + i\sin\pi)}$

b) $-6+6i = 6\sqrt{2}\left(\cos\frac{3}{4}\pi + i\sin\frac{3}{4}\pi\right)$

c) $-2i = 2\left(\cos\frac{3}{2}\pi + i\sin\frac{3}{2}\pi\right)$

d) $\sqrt{3}+i = 2\left(\cos\frac{1}{6}\pi + i\sin\frac{1}{6}\pi\right)$

e) $\sin\alpha - i\cos\alpha = \cos\left(\alpha-\frac{\pi}{2}\right) + i\sin\left(\alpha-\frac{\pi}{2}\right)$

f) $1 - i\cot\alpha = \frac{1}{\cos\left(\alpha-\frac{\pi}{2}\right)}\left(\cos\left(\alpha-\frac{\pi}{2}\right) + i\sin\left(\alpha-\frac{\pi}{2}\right)\right)$ *gdzie* $\alpha \neq k\pi$ *dla k całkowitego*

g) $1+\cos\alpha + i\sin\alpha = 2\cos\left(\frac{\alpha}{2}\right)\left(\cos\left(\frac{\alpha}{2}\right) + i\sin\left(\frac{\alpha}{2}\right)\right)$

Zadanie 4.

Oblicz wartość następujących wyrażeń. Wynik podaj w postaci algebraicznej:

a) $(1+i)^7$; b) $\left(\sqrt{3}-i\right)^{32}$; c) $(-2+2i)^8$; d) $(\cos 33° + i\cdot\sin 33°)^{10}$; e) $\left(\frac{1-i}{\sqrt{3}+i}\right)^6$;
f) $\left(-\cos\frac{\pi}{7} + i\cdot\sin\frac{\pi}{7}\right)^{14}$

Rozwiązanie

a) $(1+i)^7 = \left(\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)\right)^7 = \left(\sqrt{2}\right)^7\left(\cos\left(7\cdot\frac{\pi}{4}\right) + i\sin\left(7\cdot\frac{\pi}{4}\right)\right) =$

$$= 8\sqrt{2}\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = 8 - 8i$$

b) $\left(\sqrt{3}-i\right)^{32} = \left(2\left(\cos\frac{11}{6}\pi\right) + i\sin\frac{11}{6}\pi\right)^{32} = 2^{32}\left(\cos\left(32\cdot\frac{11}{6}\pi\right) + i\sin\left(32\cdot\frac{11}{6}\pi\right)\right) =$

$$= 4294967296\left(\cos\left(\frac{176}{3}\pi\right) + i\sin\left(\frac{176}{3}\pi\right)\right) = 4294967296\left(\cos\left(\frac{2}{3}\pi\right) + i\sin\left(\frac{2}{3}\pi\right)\right) =$$

$$= 4294967296\cdot\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = -2147483648 + 2147483648\sqrt{3}i$$

c) $(-2+2i)^8 = \left(2\sqrt{2}\left(\cos\left(\frac{3}{4}\pi\right) + i\sin\left(\frac{3}{4}\pi\right)\right)\right)^8 = \left(2\sqrt{2}\right)^8 =$

$$\left(\cos\left(8\cdot\frac{3}{4}\pi\right) + i\sin\left(8\cdot\frac{3}{4}\pi\right)\right) = 4096(\cos(6\pi) + i\sin(6\pi)) = 4096(\cos 0 + i\sin 0) =$$

$$4096$$

d) $(\cos 33° + i\cdot\sin 33°)^{10} = \cos(10\cdot 33°) + i\sin(10\cdot 33°) = \cos 330° + i\sin 330° =$

$$= \frac{\sqrt{3}}{2} - \frac{1}{2}i$$

e) $\left(\frac{1-i}{\sqrt{3}+i}\right)^6 = \left(\frac{\sqrt{2}\left(\cos\left(\frac{7}{4}\pi\right)+i\sin\left(\frac{7}{4}\pi\right)\right)}{2\left(\cos\left(\frac{1}{6}\pi\right)+i\sin\left(\frac{1}{6}\pi\right)\right)}\right)^6 = \frac{\left(\sqrt{2}\right)^6\left(\cos\left(6\cdot\frac{7}{4}\pi\right)+i\sin\left(6\cdot\frac{7}{4}\pi\right)\right)}{2^6\left(\cos\left(6\cdot\frac{1}{6}\pi\right)+i\sin\left(6\cdot\frac{1}{6}\pi\right)\right)} = \frac{8\left(\cos\left(\frac{21}{2}\pi\right)+i\sin\left(\frac{21}{2}\pi\right)\right)}{64(\cos\pi+i\sin\pi)} =$

$$= \frac{\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}}{8\cdot(-1)} = \frac{\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}}{-8} = -\frac{\sqrt{2}}{16} - \frac{\sqrt{2}}{16}i$$

f) $\left(-\cos\frac{\pi}{7} + i\cdot\sin\frac{\pi}{7}\right)^{14} = \left(\cos\left(\frac{6}{7}\pi\right) + i\sin\left(\frac{6}{7}\pi\right)\right)^{14} = \cos\left(14\cdot\frac{6}{7}\pi\right) + i\sin\left(14\cdot\frac{6}{7}\pi\right) =$

$$= \cos(12\pi) + i\sin(12\pi) = \cos 0 + i\sin 0 = 1$$

Zadanie 5.

Oblicz następujące pierwiastki:

a)$\sqrt{4i-3}$; b) $\sqrt[3]{8}$; c) $\sqrt{-2i}$; d) $\sqrt[4]{-8+8\sqrt{3}i}$; e) $\sqrt[6]{1}$

Rozwiązanie:

a) $z = 4i - 3 = -3 + 4i$

$$|z| = \sqrt{(-3)^2 + 4^2} = \sqrt{9+16} = \sqrt{25} = 5$$

$$z = -3 + 4i = 5\left(-\frac{3}{5} + \frac{4}{5}i\right)$$

ale

$$z = 5(\cos\alpha + i\sin\alpha)$$

więc

$$\sqrt{z} = \sqrt{5}\left(\cos\left(\frac{1}{2}\alpha\right) + i\sin\left(\frac{1}{2}\alpha\right)\right)$$

Skorzystamy ze wzorów

$$\sin\left(\frac{1}{2}\alpha\right) = \mp\sqrt{\frac{1-\cos\alpha}{2}}$$

I

$$\cos\left(\frac{1}{2}\alpha\right) = \mp\sqrt{\frac{1+\cos\alpha}{2}}$$

Ponieważ

$$\cos\alpha = -0{,}6$$

a

$$\sin\alpha = 0{,}8$$

więc

$$\frac{\pi}{2} < \alpha < \pi$$

ale

$$0 < \frac{\alpha}{2} < \frac{\pi}{2}$$

Oznacza to, że

$$\cos\left(\frac{1}{2}\alpha\right) = \sqrt{\frac{1+\cos\alpha}{2}} = \sqrt{\frac{1-0{,}6}{2}} = \sqrt{\frac{0{,}4}{2}} = \sqrt{0{,}2} = \sqrt{\frac{20}{100}} = \frac{1}{5}\sqrt{5} = \frac{\sqrt{5}}{5}$$

$$\sin\left(\frac{1}{2}\alpha\right) = \sqrt{\frac{1-\cos\alpha}{2}} = \sqrt{\frac{1+0{,}6}{2}} = \sqrt{\frac{1{,}6}{2}} = \sqrt{0{,}8} = \sqrt{\frac{80}{100}} = \frac{2}{5}\sqrt{5} = \frac{2\sqrt{5}}{5}$$

$$\sqrt{z} = \sqrt{-3+4i} = \sqrt{5}\left(\cos\left(\frac{1}{2}\alpha\right) + i\sin\left(\frac{1}{2}\alpha\right)\right) = \sqrt{5}\left(\frac{\sqrt{5}}{5} + \frac{2\sqrt{5}}{5}i\right) = 1+2i$$

lub

$$\sqrt{-3+4i} = -1-2i$$

b) Są 3 pierwiastki. Jeden z nich to 2, czyli w postaci trygonometrycznej

$$2 = 2(\cos 0° + i \sin 0°)$$

ponieważ

$$360°: 3 = 120°$$

Pozostałe 2, to

$$2(\cos 120° + i \sin 120°) = 2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\right) = -1 + \sqrt{3}i$$

lub

$$2(\cos 240° + i \sin 240°) = 2\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = -1 - \sqrt{3}i$$

c) $z = -2i$

$$|z| = 2$$

$$z = -2i = 2(0 - i)$$

Czyli

$$\cos\alpha = 0$$

a

$$\sin\alpha = -1$$

$$\alpha = 270°$$

$$\sqrt{z} = \sqrt{-2i} = \sqrt{2}\left(\cos\frac{270°}{2} + \sin\frac{270°}{2}\right) = \sqrt{2}(\cos 135° + i \sin 135°) = \sqrt{2}\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$$

$$= -1 + i$$

lub

$$\sqrt{z} = \sqrt{-2i} = 1 - i$$

d) $z = -8 + 8\sqrt{3}i$

$$|z| = \sqrt{(-8)^2 + \left(8\sqrt{3}\right)^2} = \sqrt{64 + 192} = \sqrt{256} = 16$$

$$z = -8 + 8\sqrt{3}i = 16\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

ale

$$z = 16(\cos\alpha + i\sin\alpha) = 16\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

czyli

$$\cos\alpha = -\frac{1}{2}$$

a

$$\sin\alpha = \frac{\sqrt{3}}{2}$$

więc

$$\alpha = 120°$$

$$\sqrt[4]{z} = \sqrt[4]{-8 + 8\sqrt{3}i} = \sqrt[4]{16(\cos 120° + i\sin 120°)} = \sqrt[4]{16}(\cos 30° + i\sin 30°)$$
$$= 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = \sqrt{3} + i$$

Pozostałe 3 pierwiastki, to

$$z = 2(\cos 120° + i\sin 120°) = 2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = -1 + \sqrt{3}i$$

$$z = 2(\cos 210° + i\sin 210°) = 2\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = -\sqrt{3} - i$$

$$z = 2(\cos 300° + i\sin 300°) = 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 1 - \sqrt{3}i$$

e) Jednym z pierwiastków jest

$$z = 1$$

Pozostałe pierwiastki, to

$$z = \cos 60° + i \sin 60° = \frac{1}{2} + \frac{\sqrt{3}}{2} i$$

$$z = \cos 120° + i \sin 120° = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$$

$$z = \cos 180° + i \sin 180° = -1$$

$$z = \cos 240° + i \sin 240° = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$$

$$z = \cos 300° + i \sin 300° = \frac{1}{2} - \frac{\sqrt{3}}{2} i$$